(1)

$$\int_0^{20\pi} \frac{1}{2+a\sin x}\,dx = 10\int_{-\pi}^{\pi} \frac{1}{2+a\sin x}\,dx = \left| \begin{array}{ll} t = \operatorname{tg}\frac{x}{2} & \sin x = \frac{2t}{1+t^2} \\ dx = \frac{2}{1+t^2}dt & \pm\pi \to \pm\infty \end{array} \right| = 10\int_{-\infty}^{\infty} \frac{1}{2+a\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

funkce je 2π-periodická

$$= 10\int_{-\infty}^{\infty} \frac{1}{1+t^2+at}\,dt = 10\int_{-\infty}^{\infty} \frac{1}{\left(t+\frac{a}{2}\right)^2 + 1 - \frac{a^2}{4}}\,dt = \frac{10}{1-\frac{a^2}{4}} \int_{-\infty}^{\infty} \frac{1}{\left(\frac{t+\frac{a}{2}}{\sqrt{1-\frac{a^2}{4}}}\right)^2 + 1}\,dt$$

$$= \left| \begin{array}{ll} u = \frac{t+\frac{a}{2}}{\sqrt{1-\frac{a^2}{4}}} & du = \frac{1}{\sqrt{1-\frac{a^2}{4}}}dt \\ \pm\infty \to \pm\infty & \end{array} \right| = \frac{10}{\sqrt{1-\frac{a^2}{4}}} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{u^2+1}\,du}_{\pi} = \frac{20\pi}{\sqrt{4-a^2}}$$

(2)

$$\int_0^{\infty} \underbrace{\sqrt{\frac{|e^{x^2} - \cos 2x - x^2|}{x^5}}}_{f(x)}\,dx$$

(a) $\int_0^1 f$ – platí

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} + o(x^4)$$

$$\cos 2x = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} + o(x^4) = 1 - 2x^2 + \frac{2}{3}x^4 + o(x^4)$$

Tedy

$$e^{-x^2} - \cos 2x - x^2 = \left(\frac{1}{2} - \frac{2}{3}\right)x^4 + o(x^4) = -\frac{1}{6}x^4 + o(x^4)$$

$x \to 0$

Celkem tedy

$$\lim_{x\to 0+} \frac{f(x)}{x^{\frac{1}{2}}} = \lim_{x\to 0+} \frac{\sqrt{\frac{\left|-\frac{1}{6}x^4 + o(x^4)\right|}{x^5}}}{x^{\frac{1}{2}}} = \lim_{x\to 0+} \sqrt{\frac{\left|-\frac{1}{6}x^4 + o(x^4)\right|}{x^4}} = \lim_{x\to 0+} \sqrt{\left|-\frac{1}{6} + \frac{o(x^4)}{x^4}\right|} = \boxed{\frac{1}{\sqrt{6}}}$$

Protože $\frac{1}{\sqrt{6}} \in \mathbb{R}\setminus\{0\}$ platí $\int_0^1 f < \infty \iff \int_0^1 \frac{1}{x^{\frac{1}{2}}} < \infty$. Tedy $\boxed{\int_0^1 f < \infty}$

(b) $\int_1^\infty f$ – platí $|e^{-x} - \cos 2x - x^2| \leq 2 + x^2$

Tedy $|f| \leq \sqrt{\frac{2+x^2}{x^5}} \overset{x \geq 1}{\leq} \sqrt{\frac{3x^2}{x^5}} \leq \sqrt{3} \cdot \frac{1}{x^{\frac{3}{2}}}$

Protože $\int_1^\infty \frac{1}{x^{\frac{3}{2}}} < \infty$ platí i $\int_1^\infty f < \infty$

Celkově $\int_0^\infty f = \int_0^1 f + \int_1^\infty f < \infty$.